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Equivariant cohomology, Fock space and loop groups

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Abstract
Equivariant de Rham cohomology is extended to the infinite-dimensional setting of a loop subgroup acting on a loop group, using Hida supersymmetric Fock space for the Weil algebra and Malliavin test forms on the loop group. The Mathai–Quillen isomorphism (in the BRST formalism of Kalkman) is defined so that the equivalence of various models of the equivariant de Rham cohomology can be established.

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1. Introduction

In this paper we show how equivariant de Rham theory may be extended to the infinite-dimensional setting. The equivariant cohomology of a manifold M which carries an action by a Lie group T is in principle the cohomology of M/T; however if the action of T has fixed points then M/T is not a manifold, and so may not have pathological cohomology. The technique for handling this situation developed by Borel [11] is to consider $(M \times E)/T$ where E is contractible and T acts freely on E. It can be shown that the cohomology of $(M \times E)/T$ does not depend on the choice of E, and that it reduces to the cohomology of M/T when the T action is free; this motivates the definition of the equivariant cohomology as the cohomology of $(M \times E)/T$. When M and T are both of finite dimension there are alternative constructions by Weil and by Cartan of real equivariant cohomology in terms of differential forms, and an equivariant de Rham theorem which establishes the equivalence of the two approaches. A full account of this work may be found in the book of Guillemin and Sternberg [19].

In this paper we construct the equivariant de Rham theory for a situation in which both the group and the manifold on which it acts are infinite dimensional, taking as manifold $\ell(H)$, the space of continuous loops in a finite-dimensional Lie group H, and as group $\ell(T)$, the group of continuous loops in a Lie subgroup T of H [34, 35]. There are a number of different but equivalent models of equivariant de Rham cohomology, including the Weil model, the Cartan model and a more recent construction by Kalkman [25] of a model (which we will refer to as the Kalkman model) which provides a direct and elegant implementation of the

Mathai–Quillen isomorphism between the Cartan model and the Weil model and other models of equivariant cohomology. We give in this paper an infinite-dimensional version of all of these constructions. We stress the difference between this equivariant cohomology and the classical equivariant cohomology of a free loop space under the natural circle action pioneered by Witten, Atiyah and Bismut [5, 10, 51]. The stochastic case is treated by Léandre [34, 35].

The work of Kalkman [25], which is further developed by Chemla and Kalkman [12], is inspired by the BRST quantization of certain topological theories in physics. The BRST construction was introduced into physics by [8, 48] as a cohomological method for handling the gauge redundancy which occurs when quantizing theories which possess symmetries. These methods involve 'ghosts' and 'antighosts' which in physicist's language are anticommuting fields while more mathematically they correspond to generators of the exterior algebra over the Lie algebra of the symmetry group and its dual. A fuller explanation of these ideas may be found in the book of Henneaux and Teitelboim [20] and the papers of Kostant and Sternberg [26] and Stasheff [47]. The BV quantization scheme [7, 16], which further develops the BRST approach, allows an extension of these techniques when the gauge symmetries are reducible, including 'ghosts for ghosts' which correspond to the even generators in the Weil algebra (see section 2) [12]. Details of these constructions in the quantum mechanical setting which corresponds to the finite-dimensional equivariant cohomology may be found in the papers of Rogers [43, 44]. The constructions in this paper are likely to be useful in providing a more rigorous analytic framework for BRST and BV methods in topological quantum field theories, although constructions of actual models, and functional integral quantization, are left for later work.

We begin, in section 2, by briefly reviewing the equivariant de Rham theory in the finite-dimensional situation, making heavy use of the book of Guillemin and Sternberg [19]. In section 3 we construct the Weil algebra for the loop group $\ell(T)$ and related operators, and in section 4 we define the notion of forms and exterior calculus on the loop group $\ell(H)$. These two sections provide us with the key ingredients for the Weil model of the equivariant cohomology, which we describe in section 5, and then in section 6 we construct the Kalkman version [25] of the Mathai–Quillen isomorphism [39] and use this to construct further models of the equivariant cohomology.

2. Equivariant de Rham theory

In this section we briefly review the equivariant de Rham theory for a finite-dimensional Lie group T acting on a finite-dimensional manifold M. The key idea is that the equivariant de Rham cohomology is the cohomology of a differential

$$\mathrm{d}_W \bigotimes \mathrm{Id} + \mathrm{Id} \bigotimes \mathrm{d}$$

acting on the 'basic' elements of $A \bigotimes \Omega(M)$ (definitions of these objects are given below) where $\Omega(M)$ is the space of forms on M and A is a W^* algebra (definition 2.1) of T with certain properties. Before defining these structures, we wish to point out the analogy here with the Borel construction $(M \times E)/T$ given above. At the algebraic level A plays the role of E, and must have properties analogous to being contractible and carrying a free T action. The restriction to basic elements is analogous to passage to the quotient.

For the finite-dimensional Lie group T the Weil algebra is defined to be

$$W(\mathfrak{t}^*) := S(\mathfrak{t}^*) \otimes \Lambda(\mathfrak{t}^*), \tag{1}$$

where $\mathfrak{t}$ is the Lie algebra of T and $\mathfrak{t}^*$ its dual while S and Λ denote the symmetric and antisymmetric tensor algebras respectively. Let $\{\xi_a | a = 1, \ldots, m\}$ be a basis of $\mathfrak{t}$ (with m

being the dimension of T) and $\{\eta^a | a = 1, \ldots, m\}$ be the dual basis of $\mathfrak{t}^*$. Denoting by u^a the generator η^a of $S(\mathfrak{t})$ and by θ^a the generator η^a of $\Lambda(\mathfrak{t})$, an element of $W(\mathfrak{t}^*)$ consists of a sum of terms of the form $u^{a_1} \hat{\otimes} \cdots \hat{\otimes} u^{a_k} \otimes \theta^{b_1} \wedge \cdots \wedge \theta^{b_l}$ where $\hat{\otimes}$ and $\wedge$ denote the symmetric and antisymmetric tensor products respectively. The algebra $W(\mathfrak{t}^*)$ is given a $\mathbb{Z}$-grading by endowing the generators u^a, θ^b with degree 2 and 1 respectively. Using this grading modulo 2 gives $W(\mathfrak{t}^*)$ the structure of a commutative super algebra.

The coadjoint representation of T on $\mathfrak{t}^*$ extends to an action of T on $W(\mathfrak{t}^*)$ by automorphisms. There is also an extension of $\mathfrak{t}$ to a super Lie algebra $\tilde{\mathfrak{t}}$ which acts on $W(\mathfrak{t}^*)$ by superderivations. The super Lie algebra $\tilde{\mathfrak{t}}$ has dimension $(m, m+1)$; its even part has a basis $\{L_a | a = 1, \ldots, m\}$ while the odd part has a basis $\{I_a | a = 1, \ldots, m\} \cup \{\mathrm{d}_W\}$. Suppose that the structure constants f_{ab}^c of $\mathfrak{t}$ in the basis $\{\xi_a | a = 1, \ldots, m\}$ are defined as usual by $[\xi_a \xi_b] = \sum_{c=1}^m f_{ab}^c \xi_c$. Then the Lie bracket of $\tilde{\mathfrak{t}}$ is defined by setting

$$\begin{aligned}
&[I_a I_b] = 0 \qquad && [L_a I_b] = \sum_{c=1}^m f_{ab}^c I_c \qquad [I_a \mathrm{d}_W] = L_a \\
&[L_a L_b] = \sum_{c=1}^m f_{ab}^c L_c \qquad && [L_a L_b] = \sum_{c=1}^m f_{ab}^c L_c \\
&[L_a \mathrm{d}_W] = 0 \qquad \text{and} && [\mathrm{d}_W \mathrm{d}_W] = 0.
\end{aligned} \tag{2}$$

Note that, because of the graded antisymmetry of the bracket, the statement $[\mathrm{d}_W \mathrm{d}_W] = 0$ is not trivial, and means that cohomology may be defined on a space where $\tilde{\mathfrak{t}}$ acts.

Since the action of $\tilde{\mathfrak{t}}$ on $W(\mathfrak{t}^*)$ is by superderivations, it is sufficient to specify the action of $\tilde{\mathfrak{t}}$ on generators:

$$\begin{aligned}
&I_a \theta^c = \delta_a^c \qquad I_a u^c = 0 \qquad L_a \theta^c = -\sum_{b=1}^m f_{ab}^c \theta^b \qquad L_a u^c = -\sum_{b=1}^m f_{ab}^c u^b \\
&\mathrm{d}_W \theta^c = u^c - \frac{1}{2} \sum_{a=1}^m \sum_{b=1}^m f_{ab}^c \theta^a \theta^b \qquad \mathrm{d}_W u^c = -\sum_{a=1}^m \sum_{b=1}^m f_{ab}^c \theta^a u^b.
\end{aligned} \tag{3}$$

It may be verified by direct calculation, using the Jacobi identity

$$\sum_{i=1}^m \left(f_{ab}^i f_{ic}^j + f_{bc}^i f_{ia}^j + f_{ca}^i f_{ib}^j \right) = 0,$$

that this does define an action of $\tilde{\mathfrak{t}}$. The degrees of the operators I_a, L_a and d_W are -1, 0 and $+1$ respectively.

A generalization of $W(\mathfrak{t}^*)$ is the concept of a W^* algebra. The definition is given in stages; further details may be found in [19].

Definition 2.1

(a) A T^* algebra is a commutative super algebra A which carries a T action ρ by automorphisms and a $\tilde{\mathfrak{t}}$ action by superderivations such that

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t} \rho(\exp(t\xi)) \bigg|_{t=0} = L_\xi, \qquad && \rho(a) L_\xi \rho(a^{-1}) = L_{\mathrm{Ad}_a \xi}, \\
&\rho(a) i_\xi \rho(a^{-1}) = i_{\mathrm{Ad}_a \xi}, && \rho(a)\, \mathrm{d} \rho(a^{-1}) = \mathrm{d}
\end{aligned} \tag{4}$$

for all ξ in $\mathfrak{t}$ and all a in T.

(b) A T^* module is a super vector space A together with a linear representation ρ of T on A and a homomorphism $\tilde{\mathfrak{t}} \to$ End A which satisfy (4).

(c) a W^* algebra of T is defined to be a T^* algebra A which is also a $W(\mathfrak{t}^*)$ module such that the map

$$W(\mathfrak{t}^*)\otimes A \to A \qquad w\times a \mapsto wa$$

is a morphism of T^* modules.

A W^* algebra A is said to be 'of type C' if it is acyclic and contains elements $\theta^a, a = 1,\dots,m$ such that $I_b\theta^a = \delta^a_b$ for all $a, b = 1,\dots,m$. An example of a W^* algebra of type C is $W(\mathfrak{t}^*)$ itself.

The action of T on M induces an action of the super Lie algebra $\tilde{\mathfrak{t}}$ on $\Omega(M)$ by superderivations: suppose that $X(\xi)$ denotes the vector field on M corresponding to the element ξ of $\mathfrak{t}$, and that $\mathcal{I}_{X(\xi)}$ denotes interior differentiation along $X(\xi)$ while $\mathcal{L}_{X(\xi)}$ denotes Lie differentiation along $X(\xi)$. Then setting I_a to act by $\mathcal{I}_{X(\xi_a)}$, L_a to act by $\mathcal{L}_{X(\xi_a)}$ and d_W by exterior differentiation d gives the required action.

Given a W^* algebra A, the basic subalgebra of $A\bigotimes\Omega(M)$, denoted by $(A\bigotimes\Omega(M))_{\mathrm{bas}}$, is defined to be the subalgebra consisting of elements λ which satisfy

$$\left(I_a\bigotimes \mathrm{Id} + \mathrm{Id}\bigotimes \mathcal{I}_a\right)\lambda = 0 \qquad \text{and} \qquad \left(L_a\bigotimes \mathrm{Id} + \mathrm{Id}\bigotimes \mathcal{L}_a\right)\lambda = 0 \tag{5}$$

for all $a = 1,\dots,m = \mathrm{Dim}\, T$. The differential $\mathrm{d}_W\bigotimes\mathrm{Id}+\mathrm{Id}\bigotimes\mathrm{d}$ restricts to an action on this subalgebra. The equivariant de Rham cohomology of M under the action of T is defined to be the cohomology of $(A\bigotimes\Omega(M))_{\mathrm{bas}}$. It can be shown that this is independent of the choice of W^* algebra A with property $\mathcal{C}$. (While $W(\mathfrak{t}^*)$ is the simplest example of such an algebra, different examples are needed to prove the equivariant de Rham theorem [19] and derive the BRST quantization for a constrained Hamiltonian system with reducible symmetries [44].)

The Weil model of the equivariant cohomology is achieved by choosing A to be $W(\tilde{\mathfrak{t}})$ itself, which is a W^* algebra with property $\mathcal{C}$. Two further models of the cohomology will now be described, the Kalkman model and the Cartan model. To do this we need to introduce an operator ψ which is the Kalkman extension [25] of the Mathai–Quillen isomorphism [39] to a linear isomorphism $(A\bigotimes\Omega(M)) \to (A\bigotimes\Omega(M))$ defined by

$$\psi = \exp\left(\sum_{a=1}^{m}\theta^a\bigotimes\mathcal{I}_a\right). \tag{6}$$

This operator is evidently both linear and invertible. Under this operation the differential d_W transforms to

$$\begin{aligned}\mathrm{d}_K &= \psi\left(\mathrm{d}_W\bigotimes\mathrm{Id}+\mathrm{Id}\bigotimes\mathrm{d}\right)\psi^{-1}\\ &= \mathrm{d}_W\bigotimes\mathrm{Id}+\mathrm{Id}\bigotimes\mathrm{d}+\sum\theta^a\bigotimes\mathcal{L}_a-\sum u^a\bigotimes\mathcal{I}_a\end{aligned} \tag{7}$$

and conditions (5) become

$$\left(\mathrm{Id}\bigotimes\mathcal{I}_a\right)\lambda = 0 \qquad \text{and} \qquad \left(L_a\bigotimes\mathrm{Id}+\mathrm{Id}\bigotimes\mathcal{L}_a\right)\lambda = 0. \tag{8}$$

Here we use the fact that if $[A, B]$ commutes with A then

$$\mathrm{e}^A B\,\mathrm{e}^{-A} = B - [B, A]\,. \tag{9}$$

Now the cohomology of d_K on the basic subalgebra defined by these new conditions is the same as that of the Weil model, and so we have an alternative model of the equivariant de Rham cohomology, the Kalkman model.

Finally we can construct the Cartan model as the cohomology of the G-invariant elements of $S(\mathfrak{t}^*)\bigotimes\Omega(M)$ with respect to the operator

$$\mathrm{d}_C = \mathrm{Id}\bigotimes\mathrm{d}-\sum u^a\bigotimes\mathcal{I}_a. \tag{10}$$

That this model is identical to the Kalkman model follows from the observation that the condition $\mathrm{Id} \bigotimes \mathcal{I}_a \lambda = 0$ is satisfied only by elements of the algebra $W \bigotimes \Omega(M)$ which are independent of all θ^a.

These various de Rham models of the equivariant cohomology relate to physical models with the so-called topological symmetry in that the BRST operators for these physical models have been shown to correspond to the differentials in the equivariant cohomology. Multinomials in the ghosts of the theory correspond to differential forms, and where the symmetry is a semi-direct product of the diffeomorphism group with another group 'ghosts for ghosts' emerge corresponding to the even generators of the Weil algebra [12, 44].

3. The Weil algebra of a loop group

In this section we construct the Weil algebra of the group $\ell(T)$ of free continuous loops in a simple simply connected Lie group T of finite dimension m.

In the case of the free loop group $\ell(T)$, one may regard the complexified Lie algebra to be the space $\ell(\mathfrak{t}_{\mathbb{C}})$ of finite energy free loops in the complexification $\mathfrak{t}_{\mathbb{C}}$ of $\mathfrak{t}$, that is, the completion of the space of smooth loops $\gamma : S_1 \to \mathfrak{t}_{\mathbb{C}}$ such that the energy

$$I(\gamma) := \int_0^1 |\gamma(s)|^2 + |\gamma'(s)|^2 \,\mathrm{d}s \tag{11}$$

is finite. (Here we are using the Killing metric on $\mathfrak{t}$ to give a Hermitian metric on the complexification $\mathfrak{t}_{\mathbb{C}}$ of the Lie algebra. It also provides us with a Hilbert space inner product on $\mathfrak{t}_{\mathbb{C}}$.) The Lie bracket of two loops in $\ell(\mathfrak{t}_{\mathbb{C}})$ is done by taking the Lie bracket in $\mathfrak{t}_{\mathbb{C}}$ of the values of the two loops in all times s.

The space $\ell(\mathfrak{t}_{\mathbb{C}})$ is a Hilbert space with inner product

$$\langle \gamma, \sigma \rangle := \int_0^1 \langle \gamma(s)^*, \sigma(s)\rangle + \langle \gamma'(s)^*, \sigma'(s)\rangle \,\mathrm{d}s. \tag{12}$$

Given an orthonormal basis $\{\xi_a : a = 1, \ldots, m\}$ of $\mathfrak{t}$, we have an orthonormal basis $\left\{\xi^T_{a,i} : a = 1, \ldots, m, i \in \mathbb{Z}\right\}$ of $\ell(\mathfrak{t}_{\mathbb{C}})$ with

$$\xi^T_{a,i}(s) = \left(\frac{1}{1 + |4\pi i|^2}\right)^{\frac{1}{2}} \mathrm{e}^{2\pi\sqrt{-1}is} \xi_a. \tag{13}$$

It will be convenient to combine i and a into a single index $A = (a_A, i_A)$ so that

$$\xi^T_A(s) = \left(\frac{1}{1 + |4\pi i_A|^2}\right)^{\frac{1}{2}} \mathrm{e}^{2\pi\sqrt{-1}i_A s} \xi_{a_A} \tag{14}$$

and the orthonormal basis is $\left\{\xi^T_A \middle| A \in \mathrm{Ind}\,\ell(\mathfrak{t})\right\}$ where $\mathrm{Ind}\,\ell(\mathfrak{t}) = \{1, \ldots, \} \times \mathbb{Z}$. If as before f^c_{ab} are the structure constants of $\mathfrak{t}$ in the basis $\{\xi_a\}$ then we can define

$$\left[\xi^T_A \xi^T_B\right] = \sum_{a_C=1}^{m} \sum_{i_C=-\infty}^{+\infty} f^{a_C}_{a_A a_B} \delta^{i_C}_{i_A+i_B} \xi^T_C \tag{15}$$

and hence give $\ell(\mathfrak{t}_{\mathbb{C}})$ the structure of a Lie algebra. We write

$$f^C_{AB} = f^{a_C}_{a_A a_B} \delta^{i_C}_{i_A+i_B} \tag{16}$$

so that

$$\left[\xi^T_A \xi^T_B\right] = \sum_{C \in \mathrm{Ind}\,\ell(\mathfrak{t})} f^C_{AB} \xi^T_C. \tag{17}$$

Using the Killing form on $\mathfrak{t}$, the loop space $\ell(\mathfrak{t}_{\mathbb{C}})$ may be identified with the space $\ell(\mathfrak{t}^*_{\mathbb{C}})$ of free finite energy loops in the dual of $\mathfrak{t}_{\mathbb{C}}$. Elements of the corresponding orthonormal basis will be written η^A_T.

Because we are now in an infinite-dimensional setting, some technical steps are required to handle the tensor products involved in the Weil algebra. Eventually we will obtain the supersymmetric Hida distribution space $W_{\infty-}(\ell(H))$ as the appropriate Weil algebra for our loop group, and show how the super extension $\widetilde{\ell(\mathfrak{t}_{\mathbb{C}})}$ of $\ell(\mathfrak{t}_{\mathbb{C}})$ acts on it. We begin by constructing the supersymmetric Fock space associated with the Hilbert space $\ell(\mathfrak{t}_{\mathbb{C}})$. Some notation is required: we define two sets of multi-indices $\mathrm{Multind}^T_s$ and $\mathrm{Multind}^T_a$. The set $\mathrm{Multind}^T_s$ consists of multi-indices

$$\underline{A}_s = \left(A_{k_1}, \ldots, A_{k_{\sharp\underline{A}_s}}\right), \tag{18}$$

where $\sharp\underline{A}_s$ denotes the number of indices in the multi-index $\underline{A}_s$ and each index A_{k_r} is in $\mathrm{Ind}\,\ell(\mathfrak{t})$ so that it is of the double form $A_{k_r} = a_{A_{k_r}}, i_{A_{k_r}}$ with $1 \leqslant a_{A_{k_r}} \leqslant m$ and $i_{A_{k_r}} \in \mathbb{Z}$; the multi-index is ordered first by $a_{A_{k_r}}$ and then by $i_{A_{k_r}}$, so that $a_{A_{k_r}} \leqslant a_{A_{k_{r+1}}}$ and if $a_{A_{k_r}} = a_{A_{k_{r+1}}}$ then $i_{A_{k_r}} \leqslant i_{A_{k_{r+1}}}$. The set $\mathrm{Multind}^T_a$ is defined similarly as the set of multi-indices

$$\underline{B}_a = \left(B_{k_1}, \ldots, B_{k_{\sharp\underline{B}_a}}\right), \tag{19}$$

except that no repeated indices B_{k_r} are allowed. The ordering rules are the same. The concatenation $\underline{A1}_s \amalg \underline{A2}_s$ of two multi-indices in $\mathrm{Multind}^T_s$ is defined as the multi-index consisting of all the indices in $\underline{A1}_s$ together with all those in $\underline{A2}_s$, rearranged in accordance with the ordering rules of $\mathrm{Multind}^T_s$. The concatenation $\underline{B1}_s \amalg \underline{B2}_s$ of two multi-indices in $\mathrm{Multind}^T_a$ is defined provided that the two multi-indices have no index in common; it is defined to be the multi-index consisting of all the indices in $\underline{B1}_s$ together with all those in $\underline{B2}_s$, rearranged in accordance with the ordering rules of $\mathrm{Multind}^T_a$. Associated with a concatenation in $\mathrm{Multind}^T_a$ is a sign $\epsilon_{B=B_1\amalg B_2}$ which is the sign of the permutation which rearranges the string of indices in $\underline{B1}_s$ followed by those in $\underline{B2}_s$ to the string of indices in $\underline{B}_s$.

Let $u^A = \eta^A_T$ be regarded as a generator of the bosonic Fock space of $\ell(\mathfrak{t}^*_{\mathbb{C}})$, and assigned degree 2, while $\theta^A = \eta^A_T$ be regarded as a generator of the fermionic Fock space of $\ell(\mathfrak{t})$, and assigned degree 1. We set

$$u^{\underline{A}_s} = u^{A_{k_1}} \hat{\otimes} \cdots \hat{\otimes} u^{A_{k_{\sharp\underline{A}_s}}} \qquad \theta^{\underline{B}_a} = \theta^{B_{k_1}} \wedge \cdots \wedge \theta^{B_{k_{\sharp\underline{B}_a}}}, \tag{20}$$

where as before $\hat{\otimes}$ denotes the symmetric and $\wedge$ the antisymmetric tensor product. The supersymmetric Fock space then consists of formal sums

$$\lambda = \sum_{\substack{\underline{A}_s \in \mathrm{Multind}^T_s \\ \underline{B}_a \in \mathrm{Multind}^T_a}} \lambda_{\underline{A}_s, \underline{B}_a} u^{\underline{A}_s} \otimes \theta^{\underline{B}_a}, \tag{21}$$

where each $\lambda_{\underline{A}_s, \underline{B}_a}$ is a complex number. For each real number r we assign weights $w_r(\underline{A}_s)$ and $w_r(\underline{B}_a)$ to the multi-indices $\underline{A}_s$ and $\underline{B}_a$ by the formulae

$$w_r(\underline{A}_s) = \prod_{A_k \in \underline{A}_s} \left(\left|i_{A_k}\right| + 1\right)^r, \qquad w_r(\underline{B}_a) = \prod_{B_k \in \underline{B}_a} \left(\left|i_{B_k}\right| + 1\right)^r \tag{22}$$

and define the Hilbert space norms when $r > 0$ by

$$\|\lambda\|_{-r,C} = \sum_{\substack{\underline{A}_s \in \mathrm{Multind}^T_s \\ \underline{B}_a \in \mathrm{Multind}^T_a}} \left|\lambda_{\underline{A}_s, \underline{B}_a}\right|^2 w_{-r}(\underline{A}_s) w_{-r}(\underline{B}_a) C^{(\sharp\underline{A}_s + \sharp\underline{B}_a)}, \tag{23}$$

(where C is a positive real number), giving us the weighted supersymmetric Fock space $(S(\ell(\mathfrak{t})) \otimes \wedge(\ell(\mathfrak{t})))_{-r,C}$. (The norms corresponding to different choices of orthonormal bases of $\mathfrak{t}$ are equivalent.)

Definition 3.1. *The set* $\cup_{r>0,C>0}(S(\ell(\mathfrak{t})) \otimes \wedge(\ell(\mathfrak{t})))_{-r,C}$ *of weighted supersymmetric Fock spaces is called the Hida supersymmetric distribution functional space and is denoted by* $(W.N.)_{-\infty}(\ell(\mathfrak{t}))$.

We refer to [21, 22, 27, 33, 36, 37, 41] for references on white noise analysis. It will now be shown that the space $W_{\infty-}(\ell(H))$ is an algebra.

Theorem 3.2. $(W.N.)_{-\infty}(\ell(\mathfrak{t}))$ *is an algebra.*

Proof. Let λ^1 and λ^2 be two elements of $(W.N.)_{-\infty}(\ell(\mathfrak{t}))$ with

$$\begin{aligned} \lambda^1 &= \sum_{\substack{\underline{A}_s \in \mathrm{Multind}_s^T \\ \underline{B}_a \in \mathrm{Multind}_a^T}} \lambda^1{}_{\underline{A}_s,\underline{B}_a} u^{\underline{A}_s} \otimes \theta^{\underline{B}_a} \\ \lambda^2 &= \sum_{\substack{\underline{A}_s \in \mathrm{Multind}_s^T \\ \underline{B}_a \in \mathrm{Multind}_a^T}} \lambda^2{}_{\underline{A}_s,\underline{B}_a} u^{\underline{A}_s} \otimes \theta^{\underline{B}_a} \end{aligned} \tag{24}$$

so that

$$\lambda^1 \cdot \lambda^2 = \sum_{\substack{\underline{A}_s \in \mathrm{Multind}_s^T \\ \underline{B}_a \in \mathrm{Multind}_a^T}} \mu_{\underline{A}_s,\underline{B}_a} u^{\underline{A}_s} \otimes \theta^{\underline{B}_a}, \tag{25}$$

where

$$\mu_{\underline{A}_s,\underline{B}_a} = \sum_{\underline{A_1}_s \amalg \underline{A_2}_s = \underline{A}_s, \underline{B_1}_s \amalg \underline{B_2}_s = \underline{B}_s} \epsilon_{B=B_1 \amalg B_2} \lambda^1{}_{\underline{A_1}_s,\underline{B_1}_a} \lambda^2{}_{\underline{A_2}_s,\underline{B_2}_a}, \tag{26}$$

and thus

$$\|\lambda^1 \cdot \lambda^2\|^2_{-r,C} = \sum_{\substack{\underline{A}_s \in \mathrm{Multind}_s^T \\ \underline{B}_a \in \mathrm{Multind}_a^T}} \left|\mu_{\underline{A}_s,\underline{B}_a}\right|^2 w_{-r}(\underline{A}_s) w_{-r}(\underline{B}_a) C^{(\sharp A + \sharp B)}. \tag{27}$$

Now, if $\underline{A_1}_s \amalg \underline{A_2}_s = \underline{A}_s$ and $\underline{B_1}_s \amalg \underline{B_2}_s = \underline{B}_s$, then

$$\begin{aligned} w_r(\underline{A}_s) &= w_r(\underline{A_1}_s) w_r(\underline{A_2}_s), & \sharp\underline{A}_s &= \sharp\underline{A_1}_s + \sharp\underline{A_2}_s, \\ w_r(\underline{B}_a) &= w_r(\underline{B_1}_a) w_r(\underline{B_2}_a) \quad \text{and} & \sharp\underline{B}_a &= \sharp\underline{B_1}_a + \sharp\underline{B_2}_a. \end{aligned} \tag{28}$$

Hence by Jensen's inequality

$$\left|\mu_{\underline{A}_s,\underline{B}_a}\right|^2 \leqslant \sum_{\underline{A_1}_s \amalg \underline{A_2}_s = \underline{A}_s, \underline{B_1}_s \amalg \underline{B_2}_s = \underline{B}_s} C_1^{(\sharp A + \sharp B)} \left|\lambda^1{}_{\underline{A_1}_s,\underline{B_1}_a}\right|^2 \left|\lambda^2{}_{\underline{A_2}_s,\underline{B_2}_a}\right|^2 \tag{29}$$

for some positive constant C_1, so that there exists a positive number C_2 such that

$$\begin{aligned} \|\lambda^1 \cdot \lambda^2\|^2_{-r,C} \leqslant &\sum_{\substack{\underline{A_1}_s \in \mathrm{Multind}_s^T, \underline{A_2}_s \in \mathrm{Multind}_s^T \\ \underline{B_1}_a \in \mathrm{Multind}_a^T, \underline{B_2}_a \in \mathrm{Multind}_a^T}} \left|\lambda^1{}_{\underline{A_1}_s,\underline{B_1}_a}\right|^2 \left|\lambda^2{}_{\underline{A_2}_s,\underline{B_2}_a}\right|^2 \\ &\times w_{-r}(\underline{A_1}_s) w_{-r}(\underline{A_2}_s) w_{-r}(\underline{B_1}_a) w_{-r}(\underline{B_2}_a) C_2^{\sharp\underline{A_1}_s + \sharp\underline{A_2}_s + \sharp\underline{B_1}_a + \sharp\underline{B_2}_a} \\ &= \|\lambda^1\|^2_{-r,C_2} \|\lambda^2\|^2_{-r,C_2} \end{aligned} \tag{30}$$

which gives the result required. □

We now define the (formal) super extension $\tilde{\mathfrak{t}}$ of the loop algebra $\ell(\mathfrak{t}_{\mathbb{C}})$ to be the algebra with even basis $\{L_A | A \in \operatorname{Ind}\ell(\mathfrak{t})\}$, and odd basis $\{L_A | A \in \operatorname{Ind}\ell(\mathfrak{t})\} \cup \{\mathrm{d}_W\}$. The brackets correspond to those in (2), with the structure constants (16). The action of $\tilde{\mathfrak{t}}$ on supersymmetric Fock space, and hence on $W_{\infty-}(\ell(H))$, will now be described; since the action is by superderivation the formal action is defined by its action on generators. First, we define the action of I_A by $I_A\theta^B = \delta_A^B$, $I_A u^B = 0$. Next we define the action of the Weil derivative d_W by

$$\begin{aligned}\mathrm{d}_W u^A &= -\sum_{B\in\operatorname{Ind}\ell(\mathfrak{h}),C\in\operatorname{Ind}\ell(\mathfrak{t})} f_{BC}^A \theta^B u^C, \qquad \text{and}\\ \mathrm{d}_W\theta^A &= u^A - \frac{1}{2}\sum_{B\in\operatorname{Ind}\ell(\mathfrak{t}),C\in\operatorname{Ind}\ell(\mathfrak{t})} f_{BC}^A \theta^B\theta^C.\end{aligned} \tag{31}$$

Since $L_A = [I_A\, \mathrm{d}_W]$ the action of L_A is determined by the action of I_A and d_W.

It is easily seen that I_A extends continuously to $W_{\infty-}(\ell(H))$. It is also the case that the Weil derivative d_W extends to $W_{\infty-}(\ell(H))$, as will be proved in the following theorem.

Theorem 3.3. d_W *extends to a superderivation of degree 1 which acts continuously on* $W_{\infty-}(\ell(H))$.

Proof. With λ as in (21) we have

$$\begin{aligned}\mathrm{d}_W\lambda &= \sum_{\substack{\underline{A}_s\in\mathrm{Multind}_s^T\\ \underline{B}_a\in\mathrm{Multind}_a^T}} \lambda_{\underline{A}_s,\underline{B}_a}\big(\mathrm{d}_W(u^{\underline{A}_s})\otimes\theta^{\underline{B}_a} + u^{\underline{A}_s}\otimes \mathrm{d}_W(\theta^{\underline{B}_a})\big)\\ &:= \mathrm{d}_W^{(1)}\lambda + \mathrm{d}_W^{(2)}\lambda.\end{aligned} \tag{32}$$

Now

$$\mathrm{d}_W u^{\underline{A}_s} = \sum_{j=1}^{\sharp\underline{A}_s} u^{A_1}\hat{\otimes}\cdots\hat{\otimes}u^{A_{j-1}}\hat{\otimes}\left(-\sum_{C,C'} f_{CC'}^{A_j}\theta^C\otimes u^{C'}\right)\hat{\otimes}u^{A_{j+1}}\hat{\otimes}\cdots\hat{\otimes}u^{A_{\sharp\underline{A}_s}} \tag{33}$$

and

$$\begin{aligned}\mathrm{d}_W\theta^{\underline{B}_a} &= \sum_{j=1}^{\sharp\underline{B}_a}(-1)^{j-1}\theta^{B_1}\wedge\cdots\wedge\theta^{B_{j-1}}\otimes u^{B_j}\otimes\theta^{B_{j+1}}\cdots\theta^{B_{\sharp\underline{B}_a}}\\ &\quad+\sum_{j=1}^{\sharp\underline{B}_s}(-1)^{j-1}\theta^{B_1}\wedge\cdots\wedge\theta^{B_{j-1}}\wedge\left(-\frac{1}{2}\sum_{C,C'} f_{CC'}^{B_j}\theta^C\wedge\theta^{C'}\right)\\ &\quad\wedge\theta^{B_{j+1}}\wedge\cdots\wedge\theta^{Bs_{\sharp\underline{B}_s}}.\end{aligned} \tag{34}$$

Also

$$\begin{aligned}\|\mathrm{d}_W(u^{\underline{A}_s})\otimes\theta^{\underline{B}_a}\|_{-r,C} &\leqslant \sum_{j=1}^{\sharp\underline{A}_s}\sum_{C,C'} C^{(\sharp\underline{A}_s+\sharp\underline{B}_a+1)}\prod_{k=1}^{j-1}\left(\left(|i_{A_k}|+1\right)^{-\frac{r}{2}}\right)\prod_{k'=j+1}^{\sharp\underline{A}_s}\left(\left(|i_{A'_k}|+1\right)^{-\frac{r}{2}}\right)\\ &\quad\times\left|f_{CC'}^{A_j}\right|(|i_C|+1)^{-\frac{r}{2}}(|i_{C'}|+1)^{-\frac{r}{2}}w_{-\frac{r}{2}}(\underline{B}_a).\end{aligned} \tag{35}$$

From (15) we know that the only contributions to the sum on the right-hand side of the above inequality occur when $i_C + i_{C'} = i_{A_j}$. Now if n is a positive integer and $r > 1$ then there exists a positive number K such that

$$\sum_{k=-\infty}^{\infty}(|k|+1)^{-r}(|n-k|+1)^{-r} \leqslant \frac{K}{(n+1)^{r-1}}, \tag{36}$$

and we deduce that there is a positive constant C_3 such that

$$\|\mathrm{d}_W(u^{\underline{A}_s})\otimes\theta^{\underline{B}_a}\|_{-r,C}\leqslant KC_3^{(\sharp\underline{A}_s+\sharp\underline{B}_a)}w_{-\frac{r}{2}+1}(\underline{A}_s)w_{-\frac{r}{2}+1}(\underline{B}_a) \tag{37}$$

if r is sufficiently large and $C_3\to C$ when $C\to 0$. Hence

$$\begin{aligned}\left\|\mathrm{d}_W^{(1)}\lambda\right\|_{r,C}&\leqslant K\sum_{\substack{\underline{A}_s\in\mathrm{Multind}_s^T\\ \underline{B}_a\in\mathrm{Multind}_a^T}}\left|\lambda_{\underline{A}_s,\underline{B}_a}\right|C_3^{(\sharp\underline{A}_s+\sharp\underline{B}_a)}w_{-\frac{r}{2}+1}(\underline{A}_s)w_{-\frac{r}{2}+1}(\underline{B}_a)\\ &\leqslant K\|\lambda\|_{-\frac{r}{2}+1,C_0}\end{aligned} \tag{38}$$

for any $C_0\to 0$ when $C\to 0$.

We now apply the Cauchy-Schwarz inequality to obtain, for some particular integer k, the standard result that

$$\sum_{\substack{\underline{A}_s\in\mathrm{Multind}_s^T\\ \underline{B}_a\in\mathrm{Multind}_a^T}}C_3^{(\sharp\underline{A}_s+\sharp\underline{B}_a)}\prod_{j=0}^{n}w_{-\frac{r}{2}+1}(\underline{A}_s)w_{-\frac{r}{2}+1}(\underline{B}_a)\leqslant\left(\prod_{n=0}^{\infty}\left(1-\frac{C_3}{n^{\frac{r}{2}-1}+1}\right)^{-1}\right)^m \tag{39}$$

which is finite. (Here as before m is the dimension of the Lie group T.) Combining this with a similar argument for d_W^2 we obtain the required result. □

Corollary 3.4. $\mathrm{d}_W{}^2=0$.

Proof. Direct calculation shows that this result is true for each generator of $W_{\infty-}(\ell(H))$; since d_W is a super derivation it must hold on all of $W_{\infty-}(\ell(H))$. □

Although we do not construct any explicit examples here other than the Weil algebra $(W.N.)_{-\infty}(\ell(\mathfrak{t}))$ itself, we remark that the concept of a W^* algebra with property $\mathcal{C}$ can be defined in our infinite-dimensional setting.

4. Malliavin calculus on a loop group

Let us recall that if Malliavin calculus had a lot of precursors, its main novelty was to complete for all the L^p existing differential operators [2, 6, 9, 21] using the tangent space to Wiener space which allows integration by parts.

Gross [18] and Airault and Malliavin [1] pointed out that it is possible to do analysis on a free loop group, because there is a tangent Hilbert space which allows integration by parts on a free loop group, via the Albeverio–Høegh-Krohn quasi-invariance formula on a loop group [3]. (See also [17].) The goal of this part is to recall, with some suitable modification, the extension of analysis on a loop group due to Fang and Franchi [14, 15] and Léandre[28–32] to include differential forms.

Let H be a compact simply connected group. Regarding the Killing form on $\mathfrak{h}$ as a Riemannian metric on $\mathfrak{h}$, we can construct Wiener measure and the associated Brownian motion on H. This measure is classically related to the heat semigroup $\exp(-t\Delta)$ of the Laplacian Δ on H, and its heat kernel $p_t(x,y)$. The Wiener measure $\mathrm{d}\mu$ on the free loop group $\ell(H)$ of maps from the S_1 into H is characterized as follows. Let $s\mapsto h_s$ be a continuous loop in the group and let $f_i, i=0,\dots,n$ be a sequence of functions from H into $\mathbb{R}$. We consider sometimes s_i, $0\leqslant s_0<s_1<\cdots<s_n$, and introduce the cylindrical function

$$F(h)=\prod_{i=0}^{n}f_i\big(h_{s_i}\big). \tag{40}$$

Then

$$\int F(h)\,\mathrm{d}\mu = \mathrm{tr}[\exp(-s_0\Delta) f_0 \exp(-(s_1 - s_0)\Delta) f_1 \cdots \exp(-(s_n - s_{n-1})\Delta) f_n \exp(-(1 - s_n)\Delta)]. \tag{41}$$

This measure lives on the continuous free loop group [23].

Just as an element ξ^H of $\mathfrak{h}$ can be regarded as a vector field on H, so can an element $\bar{\xi}^H$ of $\ell(\mathfrak{h})$, the finite energy free loop space of $\mathfrak{h}$ endowed with the Hilbert inner product and norm (as for $\ell(\mathfrak{t})$ in (11) and (12)) be regarded as a vector field on $\ell(H)$. We denote this vector field $X(\bar{\xi}^H)$; its action is given by

$$X(\bar{\xi}^H)F(g) = \frac{\mathrm{d}}{\mathrm{d}t} F\left(g\,\mathrm{e}^{t\bar{\xi}^H}\right)\Big|_{t=0}. \tag{42}$$

Albeverio and Høegh-Krohn [3] give a quasi-invariant formula which allows us to state that if $\bar{\xi}^H$ is a fixed loop in $\mathfrak{h}$ then

$$\int \langle \mathrm{d}F, X(\bar{\xi}^H)\rangle\,\mathrm{d}\mu = \int F\,\mathrm{div}X(\bar{\xi}^H)\,\mathrm{d}\mu \tag{43}$$

for all cylindrical functions F. We consider the connection $\nabla X(\bar{\xi}^H) = X(\nabla\bar{\xi}^H)$ which allows us to define higher order Sobolev spaces of the Malliavin type, so that $\nabla^r F$ belongs to all L^p spaces. If ∇F is a random element of $\ell((\mathfrak{h}\otimes\mathbb{C}))$ we can define the Sobolev norms

$$\|F\|_{r,p} = \sum_{i=1}^{r} (\mathbb{E}[\|\nabla^i F\|^p])^{\frac{1}{p}}. \tag{44}$$

We refer to [38, 40, 49, 50] for background on Malliavin calculus; see [46] for the case of a based loop group.

A k-form σ_k belongs to all the Sobolev spaces if considered as a random element of the kth order alternating product of $\ell(\mathfrak{h})\otimes\mathbb{C}$, because the tangent space of the loop group is parallelizable.

We consider forms $\sigma = \sum_{k=0}^{\infty}\sigma_k$ such that

$$\|\sigma\|_{r,p,C} = \sum_{k=0}^{\infty}\sum_{i=0}^{r} (\mathbb{E}[\|\nabla^i\sigma_k\|^p])^{\frac{1}{p}} C^k \tag{45}$$

is finite. This gives a space of forms which we denote by $W_{r,p,C}(\ell(H))$. The intersection of all these spaces is denoted by $W_{\infty-}(\ell(H))$ and called the space of Malliavin test forms of $\ell(H)$. The introduction of C is explained in the work of Connes [13] and Jones and Léandre [24].

Theorem 4.1. *$W_{\infty-}(\ell(H))$ is an algebra under the product $\wedge$.*

Proof. Let $\sigma^i = \sum_{k=0}^{\infty}\sigma_k^i$, $i = 1, 2$ be elements of $W_{\infty-}(\ell(H))$ and define $\sigma^1\wedge\sigma^2 = \sum_{k=0}^{\infty}\sigma_k^{1,2}$ where $\sigma_k^{1,2} = \sum_{k'=0}^{k}\sigma_{k'}^1\wedge\sigma_{k-k'}^2$. By the Hölder inequality

$$\mathbb{E}[\|\nabla^r(\sigma_{k'}\wedge\sigma_{k-k'})\|^p]^{\frac{1}{p}} \leqslant \left(\sum_{r'=0}^{r}\sum_{r''=0}^{r}\mathbb{E}[\|\nabla^{r'}\sigma_{k'}\|^{p_1}]^{\frac{1}{p_1}}\mathbb{E}[\|\nabla^{r''}\sigma_{k-k'}\|^{p_2}]^{\frac{1}{p_2}}\right) C_1^{k'} C_1^{(k-k')} \tag{46}$$

for some p_1 and p_2 so that

$$\|\sigma^1\wedge\sigma^2\|_{r,p,C} \leqslant C\|\sigma^1\|_{r,p_1,C_1}\|\sigma^2\|_{r,p_2,C_1} \tag{47}$$

which proves the theorem. □

Let us recall that classically the action of the exterior derivative of a k-form σ_k on $k+1$ vector fields $X_1(\bar{\xi}_1^H), \ldots, X_{k+1}(\bar{\xi}_{k+1}^H)$ is defined as follows:

$$\begin{aligned} \mathrm{d}\sigma_k\big(X_1\big(\bar{\xi}_1^H\big), \ldots, X_{k+1}\big(\bar{\xi}_{k+1}^H\big)\big) &= \sum_{i=1}^{k+1}(-1)^{i+1}\nabla\big(\sigma_k\big(X_1\big(\bar{\xi}_1^H\big), \ldots, \widehat{X_i\big(\bar{\xi}_i^H\big)}, \ldots, X_{k+1}\big(\bar{\xi}_{k+1}^H\big)\big)\big) \\ &+ \sum_{1=i<j}^{k}(-1)^{i+j}\sigma_k\big(\big[X_i\big(\bar{\xi}_i^H\big), X_j\big(\bar{\xi}_j^H\big)\big]\, X_1\big(\bar{\xi}_1^H\big), \ldots, \widehat{X_i\big(\bar{\xi}_i^H\big)} \\ &\quad , \ldots, \widehat{X_j\big(\bar{\xi}_j^H\big)}, \ldots, X_{k+1}\big(\bar{\xi}_{k+1}^H\big)\big), \end{aligned} \tag{48}$$

where $\widehat{\ }$ denotes omission of a term in a sequence. The following theorem is the main result of this section.

Theorem 4.2. *(a)* d *acts continuously on* $W_{\infty-}(\ell(H))$ *and (b)* $\mathrm{d}^2 = 0$.

Proof. (a) For fixed loops $\bar{\xi}_1^H$ and $\bar{\xi}_2^H$ the mapping

$$(\ell(\mathfrak{h}) \otimes \mathbb{C}) \times (\ell(\mathfrak{h}) \otimes \mathbb{C}) \to \ell(\mathfrak{h}) \otimes \mathbb{C}, \qquad \big(X\big(\bar{\xi}_1^H\big), X\big(\bar{\xi}_2^H\big)\big) \mapsto \big[X\big(\bar{\xi}_1^H\big), X\big(\bar{\xi}_2^H\big)\big]$$

is continuous and bilinear. We know that

$$\|\mathrm{d}\sigma_k\|_{r,p} \leqslant \|\sigma_k\|_{r+1,p} C_1^k. \tag{49}$$

Hence if $\sigma = \sum_{k=0}^{\infty}$ belongs to $W_{\infty-}(\ell(H))$ then

$$\|\mathrm{d}\sigma\|_{r,p,C} \leqslant K\|\sigma\|_{r+1,p,C_2} \tag{50}$$

for some C_2, which gives the result.

(b) The proof is exactly the same as that for finite-dimensional manifolds, given definition (48). □

We refer to [4, 42, 45] for various papers on random forms. The operation on $W_{\infty-}(\ell(H))$ of interior differentiation along $X\big(\bar{\xi}_B^H\big)$ is defined algebraically as in the finite-dimensional case and denoted $\mathcal{I}_B$; the Lie derivative along $X\big(\bar{\xi}_B^H\big)$ is denoted by $\mathcal{L}_B$ and is defined by

$$\mathcal{L}_B = [\mathcal{I}_B, \mathrm{d}]\,. \tag{51}$$

5. The Weil model of the equivariant cohomology of $\ell(H)$

In this section we construct the Weil model for the equivariant cohomology of $\ell(H)$ under the pointwise action of $\ell(T)$. As in the finite-dimensional case, this action of $\ell(T)$ on $\ell(H)$ defines an action of $\ell(T)$ on $W_{\infty-}(\ell(H))$ by algebra automorphisms. It also induces an action of the superalgebra $\widetilde{\ell(\mathfrak{t}_{\mathbb{C}})}$ on $W_{\infty-}(\ell(H))$, with I_B acting by $\mathcal{I}_B$, L_B by $\mathcal{L}_B$ and d_W by d. The $\ell(T)$ action and the $\widetilde{\ell(\mathfrak{t}_{\mathbb{C}})}$ action intertwine as in (4).

Let T be embedded in H. We wish to define the tensor product space $(W.N.)_{-\infty}(\ell(\mathfrak{t})) \bigotimes W_{\infty-}(\ell(H))$. We begin by defining the topological space $(\mathrm{W.N})_{-r_1,C_1}(\ell(\mathfrak{t})) \bigotimes W_{r,p,C}(\ell(H))$. Elements of this space have the form

$$\Xi = \sum_{\substack{\underline{A}_s \in \mathrm{Multind}_s^T \\ \underline{B}_a \in \mathrm{Multind}_a^T}} u^{\underline{A}_s} \otimes \theta^{\underline{B}_a} \bigotimes \sigma_{\underline{A}_s, \underline{B}_a}, \tag{52}$$

where

$$\|\Xi\|_{-r_1,C_1,r,p,C} := \sum_{\substack{\underline{A}_s \in \mathrm{Multind}_s^T \\ \underline{B}_a \in \mathrm{Multind}_a^T}} C_1^{(\sharp\underline{A}_s+\sharp\underline{B}_s)} w_{-r_1}(\underline{A}_s) w_{-r_1}(\underline{B}_a) \|\sigma_{\underline{A}_s,\underline{B}_a}\|_{r,p,C} \tag{53}$$

is finite. Each space $(W.N.)_{-r_1,C_1}(\ell(\mathrm{t})) \bigotimes W_{r,p,C}(\ell(H))$ is a Banach space.

Our space $(W.N.)_{-\infty}(\ell(\mathrm{t})) \bigotimes W_{\infty-}(\ell(H))$ is equal to

$$\bigcap_{r,p,c} \bigcup_{r_1,C_1} (W.N.)_{-r_1,C_1}(\ell(\mathrm{t})) \bigotimes W_{r,p,C}(\ell(H))$$

endowed with its natural topology.

Theorem 5.1. *The space $W_{\infty-}(\ell(H)) \bigotimes W_{\infty-}(\ell(H))$ is an algebra.*

Proof. Let

$$\begin{aligned}\Xi_1 &= \sum_{\substack{\underline{A}_s \in \mathrm{Multind}_s^T \\ \underline{B}_a \in \mathrm{Multind}_a^T}} u^{\underline{A}_s} \otimes \theta^{\underline{B}_a} \bigotimes \sigma^1_{\underline{A}_s,\underline{B}_a} \qquad \text{and} \\ \Xi_2 &= \sum_{\substack{\underline{A}_s \in \mathrm{Multind}_s^T \\ \underline{B}_a \in \mathrm{Multind}_a^T}} u^{\underline{A}_s} \otimes \theta^{\underline{B}_a} \bigotimes \sigma^2_{\underline{A}_s,\underline{B}_a}\end{aligned} \tag{54}$$

and define

$$\Xi_1 \cdot \Xi_2 = \sum_{\substack{\underline{A}_s \in \mathrm{Multind}_s^T \\ \underline{B}_a \in \mathrm{Multind}_a^T}} u^{\underline{A}_s} \otimes \theta^{\underline{B}_a} \bigotimes \sigma^{12}_{\underline{A}_s,\underline{B}_a} \tag{55}$$

where

$$\sigma^{12}_{\underline{A}_s,\underline{B}_a} = \sum_{\underline{A1}_s \amalg \underline{A2}_s = \underline{A}_s, \underline{B1}_s \amalg \underline{B2}_s = \underline{B}_s} \epsilon_{B=B_1 \amalg B_2} \sigma^1_{\underline{A}_s,\underline{B}_a} \sigma^2_{\underline{A}_s,\underline{B}_a} \tag{56}$$

using the concatenation conventions introduced in section 3. In each term of this equation $w_{r_1}(\underline{A1}_s) w_{r_1}(\underline{A2}_s) = w_{r_1}(\underline{A}_s)$, $w_{r_1}(\underline{B1}_a) w_{r_1}(\underline{B2}_a) = w_{r_1}(\underline{B}_a)$, $\sharp\underline{A1}_s + \sharp\underline{A2}_s = \sharp\underline{A}_s$ and $\sharp\underline{A1}_s + \sharp\underline{B2}_a = \sharp\underline{B}_a$. As a result

$$\|\sigma^1_{\underline{A}_s,\underline{B}_a} \wedge \sigma^2_{\underline{A}_s,\underline{B}_a}\|_{r,p,C} \leqslant K \|\sigma^1_{\underline{A}_s,\underline{B}_a}\|_{r,p_1,C_2} \|\sigma^2_{\underline{A}_s,\underline{B}_a}\|_{r,p_2,C_2} \tag{57}$$

and thus

$$\begin{aligned}\|\Xi_1 \cdot \Xi_2\|_{-r_1,C_1,r,p,C} \leqslant & \sum_{\underline{A1}_s \in \mathrm{Multind}_s^T, \underline{B1}_a \in \mathrm{Multind}_a^T, \underline{A2}_s \in Symind, \underline{B2}_a \in \mathrm{Multind}_a^T} C_1^{\sharp\underline{A1}_s + \sharp\underline{B1}_a} C_1^{\sharp\underline{A2}_s + \sharp\underline{B2}_a} \\ & \times w_{-r_1}(\underline{A1}_s) w_{-r_1}(\underline{B1}_a) w_{-r_1}(\underline{A2}_s) w_{-r_1}(\underline{B2}_a) \\ & \times \|\sigma^1_{\underline{A}_s,\underline{B}_a}\|_{r_1,p_1,C_2} \|\sigma^2_{\underline{A}_s,\underline{B}_a}\|_{r_2,p_2,C_2} \\ \leqslant & K \|\Xi_1\|_{-r_1,C_1,r_1,p_1,C_2} \|\Xi_2\|_{-r_1,C_1,r_1,p_2,C_2}.\end{aligned} \tag{58}$$

□

On $W_{\infty-}(\ell(H)) \bigotimes W_{\infty-}(\ell(H))$ we can define the derivative $\mathrm{d}_W \bigotimes \mathrm{Id} + \mathrm{Id} \bigotimes \mathrm{d}$.

Theorem 5.2. *The operator* $\mathrm{d}_W \bigotimes \mathrm{Id} + \mathrm{Id} \bigotimes \mathrm{d}$ *is continuous on* $W_{\infty-}(\ell(H)) \bigotimes W_{\infty-}(\ell(H))$.

Proof. We first show that $\mathrm{Id} \bigotimes \mathrm{d}$ is continuous. Suppose that Ξ is in $W_{\infty-}(\ell(H)) \bigotimes W_{\infty-}(\ell(H))$, expanded as in (52). Then

$$\left(\mathrm{Id} \bigotimes \mathrm{d}\right)\Xi = \sum_{\substack{\underline{A}_s \in \mathrm{Multind}_s^T \\ \underline{B}_a \in \mathrm{Multind}_a^T}} u^{\underline{A}_s} \otimes \theta^{\underline{B}_a} \bigotimes \mathrm{d}\sigma_{\underline{A}_s,\underline{B}_a}. \tag{59}$$

By the result of section 4,

$$\left\|\mathrm{d}\sigma_{\underline{A}_s,\underline{B}_a}\right\|_{r,p,C} \leqslant K \left\|\sigma_{\underline{A}_s,\underline{B}_a}\right\|_{r+1,p,C'} \tag{60}$$

so that

$$\left\|\left(\mathrm{Id} \bigotimes \mathrm{d}\right)\Xi\right\|_{-r_1,C_1,r,p,C} \leqslant K \|\Xi\|_{-r_1,C_1,r+1,p,C'} \tag{61}$$

and thus $\mathrm{Id} \bigotimes \mathrm{d}$ is continuous.

To show that $\mathrm{d}_W \bigotimes \mathrm{Id}$ is continuous, we recall the definition of d_W given in (31); as in section 3 we split d_W into two parts $\mathrm{d}_W \bigotimes \mathrm{Id} = \mathrm{d}_W^{(1)} \bigotimes \mathrm{Id} + \mathrm{d}_W^{(2)} \bigotimes \mathrm{Id}$. Noting that

$$\sum_{\substack{\underline{A}_s \in \mathrm{Multind}_s^T \\ \underline{B}_a \in \mathrm{Multind}_a^T}} \left\|\mathrm{d}_W^{(1)} \bigotimes \mathrm{Id}(u^{\underline{A}_s} \otimes \theta^{\underline{B}_a}) \otimes \sigma_{\underline{A}_s,\underline{B}_a}\right\|_{-r_1,C_1,r,p,C}$$
$$\leqslant \sum_{\substack{\underline{A}_s \in \mathrm{Multind}_s^T \\ \underline{B}_a \in \mathrm{Multind}_a^T}} \left(w_{1-r_1}(\underline{A}_s) w_{1-r_1}(\underline{B}_a) \tilde{C}_1^{(\sharp\underline{A}_s + \sharp\underline{B}_a)} \times \left\|\sigma_{\underline{A}_s,\underline{B}_a}\right\|_{r,p,C}\right) \tag{62}$$

where $\tilde{C}_1 \to 0$ when $C_1 \to 0$ gives

$$\left\|\left(\mathrm{d}_W^{(1)} \bigotimes \mathrm{Id}\right)\Xi\right\|_{-r_1,C_1,r,p,C} \leqslant K \|\Xi\|_{-r_1+1,\tilde{C}_1,r,p,C}. \tag{63}$$

A similar result holds for $\mathrm{d}_W^{(2)} \bigotimes \mathrm{Id}$. □

We can combine the properties of d_W and d to show that

Theorem 5.3. $(\mathrm{d}_W \bigotimes \mathrm{Id} + \mathrm{Id} \bigotimes \mathrm{d})^2 = 0$ *on* $(W.N.)_{-\infty}(\ell(\mathfrak{t})) \bigotimes W_{\infty-}(\ell(H))$

and hence we learn that $(W.N.)_{-\infty}(\ell(\mathfrak{t})) \bigotimes W_{\infty-}(\ell(H))$ is a continuous $\mathbb{Z}$-graded complex.

It is useful to note that (for any B is in $\mathrm{Ind}\,\ell(\mathfrak{t})$)

$$L_B \bigotimes \mathrm{Id} + \mathrm{Id} \bigotimes \mathcal{L}_B = \left[\mathrm{d}_W \bigotimes \mathrm{Id} + \mathrm{Id} \bigotimes \mathrm{d},\, I_B \bigotimes \mathrm{Id} + \mathrm{Id} \bigotimes \mathcal{I}_B\right]. \tag{64}$$

As a result, if we define $((W.N.)_{-\infty}(\ell(\mathfrak{t})) \bigotimes W_{\infty-}(\ell(H)))_{\mathrm{bas}}$ to be the subalgebra of $W_{\infty-}(\ell(H)) \bigotimes W_{\infty-}(\ell(H))$ whose elements Ξ satisfy

$$\begin{aligned} &\left(L_B \bigotimes \mathrm{Id} + \mathrm{Id} \bigotimes \mathcal{L}_B\right)\Xi = 0 \qquad \text{and} \\ &\left(I_B \bigotimes \mathrm{Id} + \mathrm{Id} \bigotimes \mathcal{I}_B\right)\Xi = 0 \qquad \forall\, B \in \mathrm{Ind}\,\ell(\mathfrak{t}) \end{aligned} \tag{65}$$

then

$$\begin{aligned} &\left(\mathrm{d}_W \bigotimes \mathrm{Id} + \mathrm{Id} \bigotimes \mathrm{d}\right)\left((W.N.)_{-\infty}(\ell(\mathfrak{t})) \bigotimes W_{\infty-}(\ell(H))\right)_{\mathrm{bas}} \\ &\subset \left((W.N.)_{-\infty}(\ell(\mathfrak{t})) \bigotimes W_{\infty-}(\ell(H))\right)_{\mathrm{bas}}. \end{aligned} \tag{66}$$

This allows us to define the Weil model of the equivariant cohomology of $\ell(H)$:

Definition 5.4. *The Weil model of the equivariant cohomology of* $\ell(H)$ *is the cohomology of* $\mathrm{d}_W \bigotimes \mathrm{Id} + \mathrm{Id} \bigotimes \mathrm{d}$ *acting on*

$$\big((W.N.)_{-\infty}(\ell(\mathfrak{t})) \bigotimes W_{\infty-}(\ell(H))\big)_{\mathrm{bas}}.$$

Given a more general W^* algebra A for $\ell(T)$ with property $\mathcal{C}$ one can show that the cohomology of $(A \bigotimes W_{\infty-}(\ell(H)))_{\mathrm{bas}}$ is independent of the choice of A. The arguments to prove this are the same as those in the finite-dimensional case, being algebraic rather than analytic.

In the following section we show how two further models of this cohomology may be constructed.

6. The Kalkman model and the Cartan model of the equivariant cohomology of $\ell(H)$

From the algebraic point of view, the constructions in this section of the Kalkman model and the Cartan model are the same as those given by Kalkman in [25]. We use an isomorphism Ψ of $(W.N.)_{-\infty}(\ell(\mathfrak{t})) \bigotimes W_{\infty-}(\ell(H))$ which has the formal expression $\exp\big(\sum_{B\in \mathrm{Ind}\,\ell(\mathfrak{t})} \theta^B \bigotimes \mathcal{I}_B\big)$ to interpolate between the Weil model and the Kalkman model, and to derive the Cartan model. However in the infinite-dimensional setting of this paper we must define the operator Ψ in a form where it can be shown that it acts continuously on $W_{\infty-}(\ell(H)) \bigotimes W_{\infty-}(\ell(H))$.

We begin by observing that since T is a closed Lie subgroup of H, we may extend the orthonormal basis $\{\xi_a | a = 1, \ldots, m = \mathrm{Dim}\, T\}$ of $\mathfrak{t}$ to an orthonormal basis $\{\xi_a | a = 1, \ldots, n = \mathrm{Dim}\,\mathrm{H}\}$ of $\mathfrak{h}$; the basis $\big\{\xi^T_{a,i} : a = 1, \ldots, m, i \in \mathbb{Z}\big\}$ of $\ell(\mathfrak{t})$ may similarly be expanded to a basis $\big\{\xi^H_{a,i} : a = 1, \ldots, n, i \in \mathbb{Z}\big\}$. A typical form σ on $\ell(H)$ may be expanded as

$$\sigma = \sum_{C \in \mathrm{Multind}^H_a} \sigma_{\underline{C}_a} \omega_H^{\underline{C}_a}, \tag{67}$$

where $\big\{\omega^B_H \big| B \in \mathrm{Ind}\,\ell(\mathfrak{h})\big\}$ is the dual basis to $\big\{\xi^H_{a,i} : a = 1, \ldots, n, i \in \mathbb{Z}\big\}$, so that

$$\|\sigma\|_{r,p,C} = \sum_{r'=0}^{r} \sum_{k=0}^{\infty} \left(\mathbb{E}\left[\left(\sum_{\underline{B}_a \in \mathrm{Multind}^H_a, \sharp \underline{B}_a = k} \big\| \nabla^{r'} \sigma_{\underline{B}_a} \big\|^2 \right)^{\frac{p}{2}} \right]^{\frac{1}{p}} C^k \right). \tag{68}$$

Now define the operator Ψ on $(W.N.)_{-\infty}(\ell(\mathfrak{t})) \bigotimes W_{\infty-}(\ell(H))$ by

$$\Psi = \sum_{B \in \mathrm{Multind}^T_a} \Psi_{\underline{B}_a}, \tag{69}$$

where

$$\Psi_{\underline{B}_a} = \prod_{j=1}^{\sharp \underline{B}_a} \theta^{B_j} \bigotimes \mathcal{I}_{B_j}. \tag{70}$$

Also let Ξ belong to $W_{\infty-}(\ell(H)) \bigotimes W_{\infty-}(\ell(H))$ with

$$\Xi = \sum_{\underline{A}_s \in \mathrm{Multind}^T_s, \underline{B}_s \in \mathrm{Multind}^T_a} u^{\underline{A}_s} \otimes \theta^{\underline{B}_a} \bigotimes \sigma_{\underline{A}_s, \underline{B}_a} \tag{71}$$

as before.

Expanding Ξ further using the basis of forms ω_H^B on $\ell(H)$ gives

$$\Xi = \sum_{\substack{\underline{A}_s \in \mathrm{Multind}_s^T, \underline{B}_s \in \mathrm{Multind}_a^T \\ \underline{C}_s \in \mathrm{Multind}_a^H}} u^{\underline{A}_s} \otimes \theta^{\underline{B}_a} \bigotimes \sigma_{\underline{A}_s, \underline{B}_a, \underline{C}_a} \omega_H^{\underline{C}_a}. \tag{72}$$

so that

$$\Psi\Xi = \sum_{D \in \mathrm{Multind}_a^T} \sum_{\substack{\underline{A}_s \in \mathrm{Multind}_s^T, \underline{B}_s \in \mathrm{Multind}_a^T \\ \underline{C}_s \in \mathrm{Multind}_a^H}} (-1)^{\sharp \underline{B}_a \sharp \underline{D}_a} u^{\underline{A}_s} \otimes \theta^{\underline{D}_a} \otimes \theta^{\underline{B}_a} \bigotimes \sigma_{\underline{A}_s, \underline{B}_a, \underline{C}_a} \mathcal{I}_{\underline{D}_a} \omega_H^{\underline{C}_a}. \tag{73}$$

Hence

$$\begin{aligned} \|\Psi\Xi\|_{-r_1, C_1, r, p, C} \leqslant & \sum_{\underline{A}_s \in \mathrm{Multind}_s^T, \underline{B}_s \in \mathrm{Multind}_a^T, D \in \mathrm{Multind}_a^T} C_1^{\sharp \underline{A}_s + \sharp \underline{B}_a + \sharp \underline{D}_a} w_{-r_1}(\underline{A}_s) w_{-r_1}(\underline{B}_a) w_{-r_1}(\underline{D}_a) \\ & \times \sum_{r'=0}^{r} \sum_{k=0}^{\infty} \left(\mathbb{E}\left[\left(\sum_{\underline{C}_a \in \mathrm{Multind}_a^H, \underline{D}_a \subset \underline{C}_a, \sharp \underline{C}_a - \sharp \underline{D}_a = k} \left\| \nabla^{r'} \sigma_{\underline{A}_s, \underline{B}_a, \underline{C}_a} \right\|^2 \right)^{\frac{p}{2}} \right]^{\frac{1}{p}} C^k \right). \end{aligned} \tag{74}$$

We observe that

$$\begin{aligned} & \sum_{r'=0}^{r} \sum_{k=0}^{\infty} \left(\mathbb{E}\left[\left(\sum_{\underline{C}_a \in \mathrm{Multind}_a^H, \underline{D}_a \subset \underline{C}_a, \sharp \underline{C}_a - \sharp \underline{D}_a = k} \left\| \nabla^{r'} \sigma_{\underline{A}_s, \underline{B}_a, \underline{C}_a} \right\|^2 \right)^{\frac{p}{2}} \right]^{\frac{1}{p}} C^k \right) \\ & \quad \leqslant \sum_{r'=0}^{r} \sum_{k=0}^{\infty} \left(\mathbb{E}\left[\left(\sum_{\underline{C}_a \in \mathrm{Multind}_a^H, \sharp \underline{C}_a = k + \sharp \underline{D}_a} \left\| \nabla^{r'} \sigma_{\underline{A}_s, \underline{B}_a, \underline{C}_a} \right\|^2 \right)^{\frac{p}{2}} \right]^{\frac{1}{p}} C^{k + \sharp \underline{D}_a} \right) \end{aligned} \tag{75}$$

if $C > 1$ which we can take to be the case. This will assist us to prove the following theorem.

Theorem 6.1. *The operator Ψ is a continuous linear automorphism of $(W.N.)_{-\infty}(\ell(\mathfrak{t})) \bigotimes W_{\infty-}(\ell(H))$.*

Proof. We first show that Ψ is continuous.

$$\|\Psi\Xi\|_{-r_1, C_1, r, p, C} \leqslant \sum K w_{-r_1}(\tilde{\underline{B}}_a) \|\Xi\|_{-r_1, C_1, r, p, C} \tag{76}$$

However if r_1 is big enough

$$\sum_{\underline{B}_a \in \mathrm{Multind}_a^T} w_{-r_1}(\underline{B}_a) \leqslant \left(\prod_{l=1}^{\infty} \left(1 - \frac{1}{|l|^{r_1} + 1} \right)^{-1} \right)^m. \tag{77}$$

Clearly Ψ is linear. To show that Ψ is an automorphism, we note that formally

$$\Psi = \prod_{B \in \mathrm{Ind}\,\ell(\mathfrak{t})} \left(1 + \theta^B \bigotimes \mathcal{I}_B\right) = \exp\left(\sum_{B \in \mathrm{Ind}\,\ell(\mathfrak{t})} \theta^B \bigotimes \mathcal{I}_B \right) \tag{78}$$

which has formal inverse

$$\Psi' = \exp\left(\sum_{B\in \mathrm{Ind}\,\ell(\mathrm{t})} -\theta^B \bigotimes \mathcal{I}_B\right). \tag{79}$$

Hence Ψ has a continuous inverse Ψ^{-1} where

$$\Psi_{\underline{B}_a} = \prod_{j=1}^{\sharp \underline{B}_a} (-1)^{\sharp \underline{B}_a} \theta^{B_j} \bigotimes \mathcal{I}_{B_j}. \tag{80}$$

□

The construction of the operator Ψ allows us to prove that there is a second differential on $(W.N.)_{-\infty}(\ell(\mathrm{t})) \otimes W_{\infty-}(\ell(H))$, which we will denote by d_K, with isomorphic cohomology. This is the content of the following theorem, which shows that in this infinite-dimensional setting the Kalkman or BRST model [25] of the cohomology can be constructed. In a final theorem we show (as in [25] in the finite-dimensional case) that there is an analogue of the Cartan model.

Theorem 6.2

(a) $\mathrm{d}_K = \mathrm{d}_W \bigotimes \mathrm{Id} + \mathrm{Id} \bigotimes \mathrm{d} + \sum \theta^A \bigotimes \mathcal{L}_A - \sum u^A \bigotimes \mathcal{I}_A$ *is a continuous operator on* $(W.N.)_{-\infty}(\ell(\mathrm{t})) \otimes W_{\infty-}(\ell(H))$

(b) $\mathrm{d}_K^2 = 0$.

(c) Ψ *is an isomorphism of the cohomology of* $\mathrm{d}_W \bigotimes \mathrm{Id} + \mathrm{Id} \bigotimes \mathrm{d}$ *acting on* $((W.N.)_{-\infty}(\ell(\mathrm{t})) \bigotimes W_{\infty-}(\ell(H)))_{\mathrm{bas}}$ *with the cohomology* d_K *acting on the subspace of* $(W.N.)_{-\infty}(\ell(\mathrm{t})) \bigotimes W_{\infty-}(\ell(H))$ *consisting of elements* Ξ *which satisfy*

$$\left(L_B \bigotimes \mathrm{Id} + \mathrm{Id} \bigotimes \mathcal{L}_B\right)\Xi = 0 \qquad \text{and} \qquad \left(I_B \bigotimes \mathrm{Id}\right)\Xi = 0 \quad \forall B \in \mathrm{Ind}\,\ell(\mathrm{t}). \tag{81}$$

Proof. The argument here is purely algebraic and is the same as that given in section 2 for the finite-dimensional case. □

We can also use the arguments in section to establish the equivalence of the Cartan model:

Corollary 6.3. *Suppose that* $(W.N.)_{-\infty}(\ell(\mathrm{t}))_b$ *is the algebra* $(W.N.)_{-\infty}(\ell(\mathrm{t}))$ *with all odd generators set to zero. Then the cohomology of the subalgebra* $((W.N.)_{-\infty}(\ell(\mathrm{t}))_b \bigotimes W_{\infty-}(\ell(H)))^G$ *of* $((W.N.)_{-\infty}(\ell(\mathrm{t}))_b \bigotimes W_{\infty-}(\ell(H)))$ *which consists of elements which are annihilated by* $L_B \bigotimes \mathrm{Id} + \mathrm{Id} \bigotimes \mathcal{L}_B$ *with respect to the differential* $\mathrm{d}_C = \mathrm{Id} \bigotimes \mathrm{d} - \sum u^A \bigotimes \mathcal{I}_A$ *is isomorphic to the equivariant cohomology of* $\ell(H)$ *under the* $\ell(T)$ *action.*

Finally we observe that provided that a suitable tensor product can be defined the arguments used in the finite-dimensional case to show that the basic cohomology of $A \bigotimes \Omega(M)$ is independent of the choice of W^* algebra A would also be valid in our loop group setting, and might make it possible to prove an equivariant de Rham theorem.

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